Ontario

Ministry of
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GRADE 10 ADVANCED GEOMETRY

NOTES FOR TEACHERS

CONTENTS

(Notes for G1, G4, G5, and G6 are under preparation and will be distributed later.)

The resource notes in this module are related to the Grade 10 Advanced Geometry Strand for Intermediate Division Mathematics 1977, Draft Copy. They are intended for use by teachers and board curriculum committees as they plan the mathematics program for their schools.

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Ministry of Education Ontario

GRADE 10 ADVANCED GEOMETRY

SECTION 2: COORDINATES AND TRANSFORMATIONS

RELATED SECTIONS AND TOPICS

PAST	FY:	Pages 11, 12
	Ed PJ Div:	Pages 66, 74-76
	Gr 7:	N 6; A 1b; A 3b; G 1b; G 2a; G 4ab; G 5ab
	Gr 8:	N 3a; N 4ab; A 2b; G 2ac; G 4abc
	Gr 9 Gen:	N 2c; N 6b; A 4; G 2; G 3; G 4abc
	Gr 9 Adv:	N 2e; N 5c; G 2; G 3; G 5
PRESENT	Gr 10 Gen:	N 2cde; A 1; G 2
	Gr 10 Adv:	A 1bg; A 2; G 1; G 3ce; G 5

COORDINATES AND TRANSFORMATIONS

Introduction

The students' earlier experiences in representing transformations have been essentially on non-coordinate grids and on plain paper. See the notes for 7G 4, 8G 2abc, 8G 4ab, 9Gen G 3, 9Gen 4ad, and 9Adv G 3abcd. Transformations on a coordinate grid are implied in the earlier work with geoboards and dot paper, and made more explicit in the notes for 9Adv G 2b. See also the notes for 9Gen G 2b and 9Gen G 4a.

In this section, the ideas from the earlier sections are extended by showing that transformations can be described by mapping rules. Then some of the properties of the transformations are checked for numerical cases to show the consistency between the mapping rules and the fundamental properties. At the optional level, some of the properties introduced in Section G 1 are re-examined using the mapping rules to develop them, and 2 x 2 matrices are introduced for representing special cases of reflection and rotation, and dilatations with centre the origin.

Until now, we have been primarily concerned with a figure and its image under the transformations studied, and with the properties of the figures that are invariant and those that change. We have not attempted to show consciously that a transformation is a 1 - 1 mapping of the whole plane onto itself. The ideas that have been investigated to this stage have involved only local applications of the transformations.

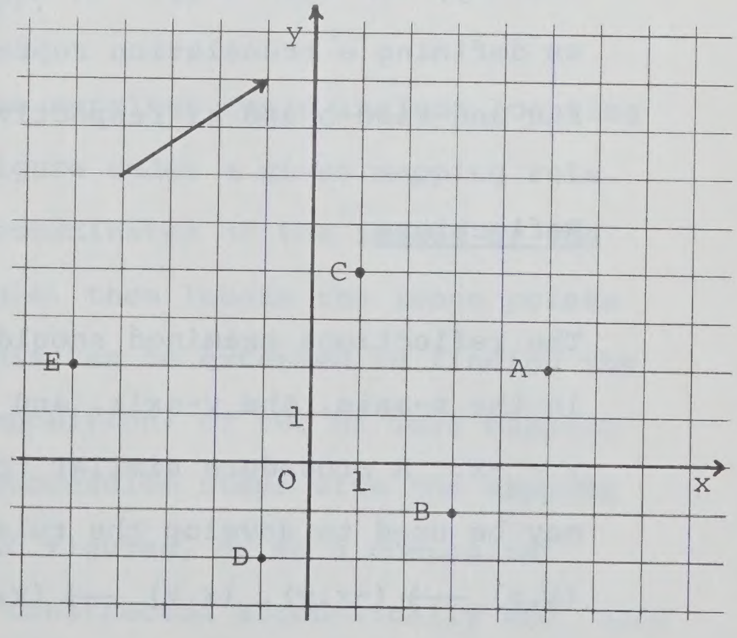
Once we begin to use mapping rules, it is apparent that every point in the plane (x,y) can be related to a unique image point (x',y'). It is now implicit that the transformations apply to the whole plane. However, this is not the purpose of this section, nor should it be stressed at this time. Such notions should occur naturally when opportune situations arise.

a) <u>Developing mapping rules for the image of a point (x,y) under translation, special cases of reflection and rotation, and dilatation</u>

The students should locate image points on a coordinate grid, using the fundamental property of each transformation. The coordinates of the given points and their images can be noted and compared. From these investigations, specific numerical rules can be stated, and then generalized to the mapping rule for the transformation.

<u>Translation</u>

For example, the students could be asked to locate the translation image of each of the points marked, using the given arrow; then to write the co-ordinates of the image points.



They might be asked the following questions.

"How did you locate C_1?"

"What are its coordinates?"

"How did you find them?"

(The student might say, "I read them from the grid". In which case, "How are they related to the coordinates of C?") The student might say, "Right 3, up 2; add 3 to 1, 2 to 4 to get (4,6)".

This procedure can be repeated for other points, and then the rule for this translation described as $(x,y) \longrightarrow (x + 3, y + 2)$. To which points does this rule apply? Try some, first points on the grid such as $(0,0) \longrightarrow (\ , \ )$, $(-5,3) \longrightarrow (\ , \ )$, then try $(473,216) \longrightarrow (\ \ , \ \ )$ to show that the rule extends beyond the immediate view of the plane.

The above method can be repeated for a few other numerical cases of translations, introducing their mapping rules. Finally, the rule $(x,y) \longrightarrow (x + h, y + k)$ can be introduced as defining a translation represented by the arrow with run and rise h and k, respectively.

Reflections

The reflections examined should be restricted to reflection in the x-axis, the y-axis, and the line $y = x$, and optionally $y = -x$. A procedure similar to the one above for translations may be used to develop the rules, respectively $(x,y) \longrightarrow (x,-y)$, $(x,y) \longrightarrow (-x,y)$, $(x,y) \longrightarrow (y,x)$, and $(x,y) \longrightarrow (-y,-x)$. These rules need not be memorized; they can be reconstructed from numerical cases.

Rotations

The rotations studied should be restricted to those with centre the origin, and measures 90°, 180°, and 270° (-90°). Students can locate the image points using tracing paper or a visual method. Then a line of reasoning similar to the one above can eventually lead to the respective mapping rules $(x,y) \longrightarrow (-y,x)$, $(x,y) \longrightarrow (-x,-y)$, and $(x,y) \longrightarrow (-y,-x)$. These rules need not be memorized; they can be reconstructed from numerical cases.

In order to determine mapping rules for rotations with centre the origin and of measure θ, the student would need to know trigonometry related to the unit circle and, ideally, the use of matrices to represent transformations.

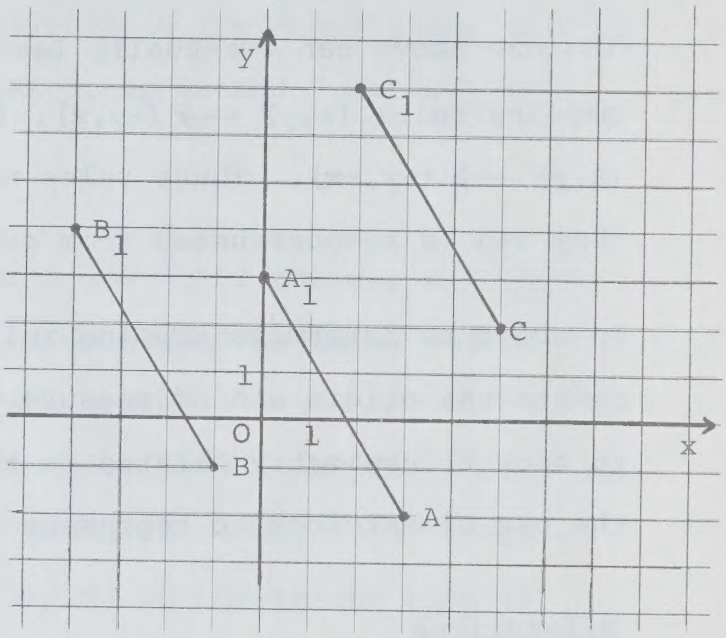
Dilatations

Dilatation images should be examined for cases with centre the origin, with any numerical value for the scale factor. The rule for a dilatation with centre the origin and scale factor k is $(x,y) \longrightarrow (kx, ky)$.

Exercises related to the above mappings could include locating the image of a rectilinear figure under a given mapping rule. The student should find the coordinates of the images of the vertices using the mapping rule, then locate the image points and draw the image figure. This can be extended to finding the image under a succession (composition) of two or more mapping rules. (This can be done as successive steps with the mapping rules applied to the geometric figures, or as a composite mapping in which the rule is constructed algebraically and "gets the whole job done in one step" — the latter approach could be left to the students to discover for themselves.)

Finally, the students should check for numerical cases that the fundamental property (defining property) of each transformation is upheld by its mapping rule. Two examples follow.

Example 1



Using the mapping rule $(x,y) \longrightarrow (x - 3, y + 5)$ the student could select any three points, say $A(3,-2)$, $B(-1,-1)$, and $C(5,2)$

Then the images under the rule can be found:

$A_1(0,3)$, $B_1(-4,4)$, and $C_1(2,7)$.

Now the slopes and lengths of the segments AA_1, BB_1, and CC_1 can be found.

$$\text{Slope } AA_1 = \frac{3-(-2)}{0-3} = \frac{5}{-3} \qquad \text{Slope } BB_1 = \frac{4-(-1)}{-4-(-1)} = \frac{5}{-3} \qquad \text{Slope } CC_1 = \frac{7-2}{2-5} = \frac{5}{-3}$$

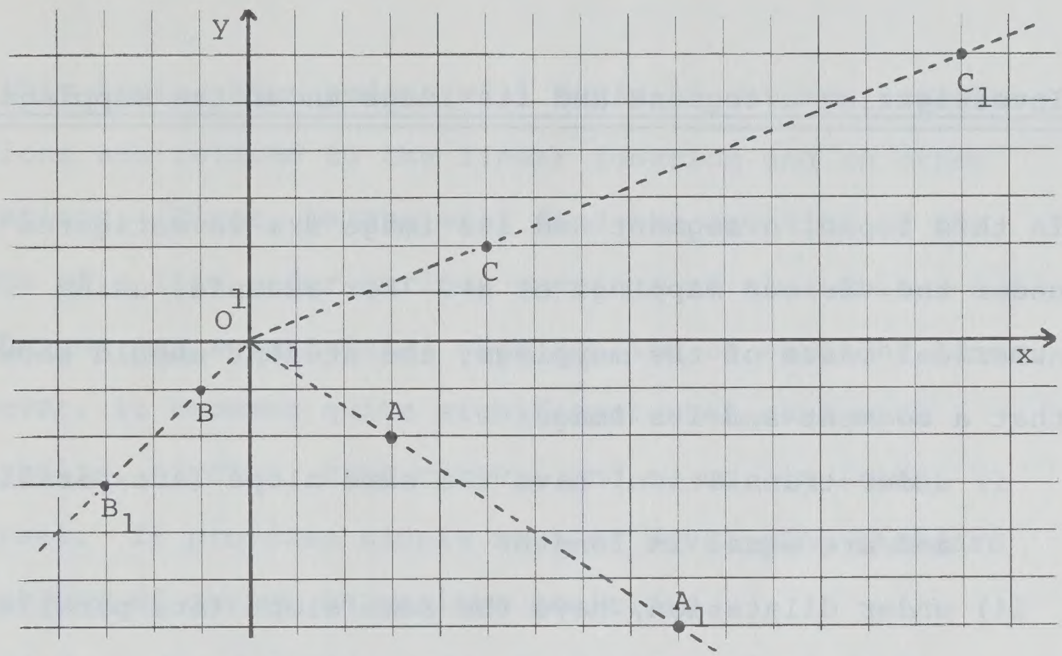
$$AA_1 = \sqrt{(3)^2 + (-2-3)^2} = \sqrt{34} \qquad BB_1 = \sqrt{(-4+1)^2 + (4+1)^2} = \sqrt{34}$$

$$CC_1 = \sqrt{(2-5)^2 + (7-2)^2} = \sqrt{34}$$

The above shows that this mapping rule defines a translation, since $AA_1 = BB_1 = CC_1$ and $AA_1 \parallel BB_1 \parallel CC_1$

Example 2

The images of A, B, and C under $(x,y) \longrightarrow (3x,3y)$ are $A_1(9,-6)$, $B_1(-3,-3)$, and $C_1(15,6)$.



Slope OA $= \frac{-2}{3}$

Slope $OA_1 = \frac{-6}{9}$

$= \frac{-2}{3}$

Slope OB $= \frac{-1}{-1}$

$= 1$

Slope $OB_1 = \frac{-3}{-3}$

$= 1$

Slope OC $= \frac{2}{5}$

Slope $OC_1 = \frac{6}{15}$

$= \frac{2}{5}$

Therefore O, A, A_1 are collinear, O, B, B_1 are collinear, and O, C, C_1 are collinear.

$OA = \sqrt{9 + 4}$

$= \sqrt{13}$

$OA_1 = \sqrt{81 + 36}$

$= \sqrt{117}$

$= \sqrt{9 \times 13}$

$= 3\sqrt{13}$

$OB = \sqrt{1 + 1}$

$= \sqrt{2}$

$OB_1 = \sqrt{9 + 9}$

$= \sqrt{18}$

$= \sqrt{9 \times 2}$

$= 3\sqrt{2}$

$OC = \sqrt{25 + 4}$

$= \sqrt{29}$

$OC_1 = \sqrt{225 + 36}$

$= \sqrt{261}$

$= \sqrt{9 \times 29}$

$= 3\sqrt{29}$

Therefore $OA_1:OA = 3:1$; $OB_1:OB = 3:1$; and $OC_1:OC = 3:1$.

The above shows that the mapping rule defines a dilatation with centre O and scale factor 3, for the points tested.

The above examples give the students lots of practice with slope, distance, collinearity, the defining properties of the transformations, and help in consolidating basic skills with integers, squaring, radicals, and fractions.

b) <u>Investigating a segment and its image under the mappings in a)</u>

In this topic, a segment and its image are investigated under the various mappings of a). For example, using numerical cases of the mappings, the student should show that a segment and its image:

i) under translation, have the same slope (are parallel) and are equal in length;

ii) under dilatation, have the same slope (are parallel) and their lengths are in the ratio 1:k, where k is the scale factor;

iii) under a rotation of 90°, or 270°, have negative reciprocal slopes (are perpendicular) and are equal in length;

iv) under a rotation of 180°, have the same slope (are parallel) and are equal in length;

v) under a reflection in the x- or y-axis, have opposite slopes (example, 2/3 and -2/3) and are equal in length;

vi) under a reflection in y = x or y = -x, have reciprocal slopes (example, 2/3 and 3/2) and are equal in length.

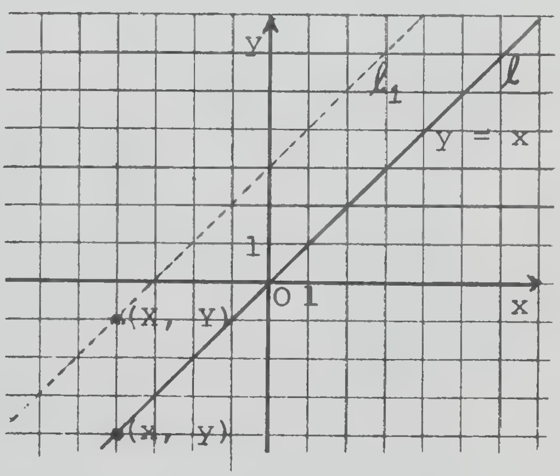
c) Determining the equation of the image of a line under the mappings in a)

See notes for teachers 9Adv G 2e.

In this topic, the student will begin to see how transformations are related to the linear function and to other relations. First, he/she will find the equation of the image of a line under various mappings of a). This technique may seem trivial when applied to straight lines. However, it becomes quite significant when used with parabolas, circles, sine curves, and so on, in later courses. It provides simple ways of developing standard equations of curves in various positions.

Example 1

Find the equation of the image of ℓ : $y = x$ under the mapping $(x,y) \longrightarrow (x, y + 3)$.



The mapping rule describes a vertical translation of +3 units; ℓ_1 can be drawn as shown.

Let (X,Y) represent any point in the image line

Then X = x and Y = y + 3

i.e. y = Y - 3

By substituting for x and y in y = x we obtain

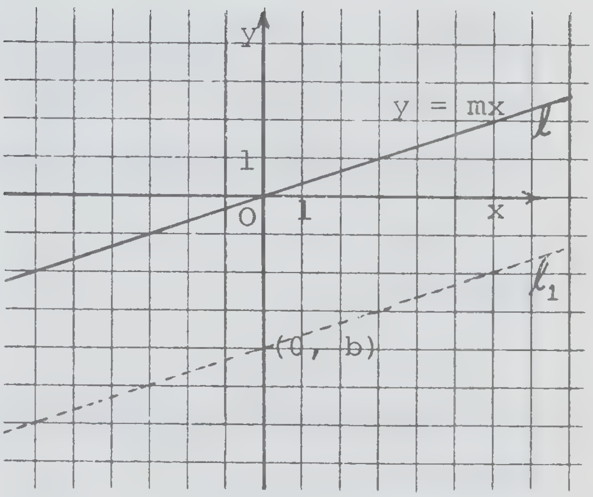
$$Y - 3 = X$$

$$\therefore \quad Y = X + 3$$

This equation defines the line ℓ_1: y = x + 3

Example 2

Find the equation of the image of line y = mx under the mapping (x,y) ⟶ (x, y + b).



The image line ℓ_1 is parallel to ℓ.
Let (X,Y) be any point in ℓ_1. Then X = x,
and Y = y + b
i.e. y = Y - b

Substituting for x and y in y = mx, we obtain the equation of ℓ_1.

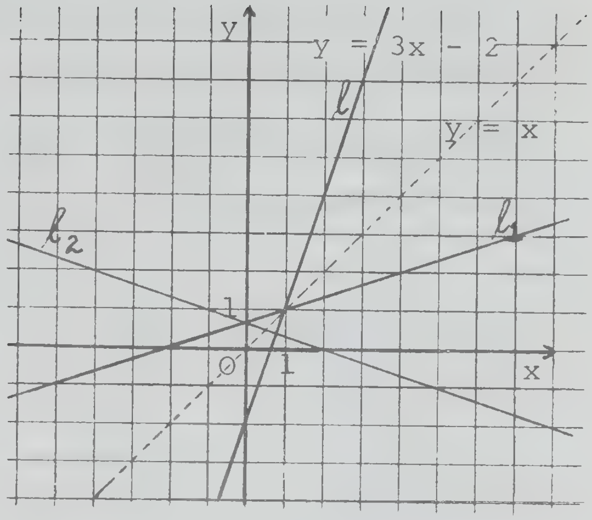
$$Y - b = mX$$

$$\therefore \quad y = mx + b$$

Example 3

i) Find the equation of ℓ_1, the image of ℓ: $y = 3x - 2$ under reflection in $y = x$.

ii) Find the equation of ℓ_2, the image of ℓ_1 under reflection in the y-axis.



i) Let (X,Y) represent any point in ℓ_1. The mapping rule for reflection in $y = x$ is $(x,y) \longrightarrow (y,x)$.
Thus $X = y$ and $Y = x$. Substituting in $y = 3x - 2$, the equation of ℓ_1 is:

$$X = 3Y - 2$$

$$\therefore \quad 3Y = X + 2$$

$$\therefore \quad y = \frac{x}{3} + \frac{2}{3}$$

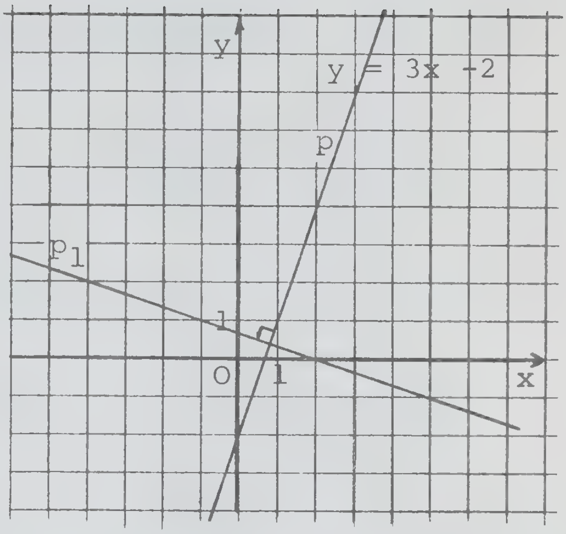
ii) Let (X,Y) represent any point in ℓ_2.
The mapping rule for reflection in the y-axis is
$(x,y) \longrightarrow (-x,y)$.
Thus $X = -x$ and $Y = y$
i.e. $x = -X$
Substituting for x and y in the equation of   ,
the equation of ℓ_2 is $\quad Y = \frac{-X}{3} + \frac{2}{3}$

$$\therefore \quad y = -\frac{x}{3} + \frac{2}{3}$$

Compare the slope of ℓ_2 with the slope of ℓ. They are negative reciprocals; thus $\ell_2 \perp \ell$.

Example 4

Find the equation of p_1, the image of p: $y = 3x - 2$ under rotation of 90° with centre the origin.



The mapping rule is

$(x,y) \longrightarrow (-y,x)$.

Let (X,Y) represent any point in p_1.

Then $X = -y$ and $Y = x$

i.e. $y = -X$ $x = Y$

Substituting for y and x in $y = 3x - 2$, the equation of p_1 is:

$$-X = 3Y - 2$$

$$\therefore \quad 3Y - 2 = -X$$

$$\therefore \quad 3Y = -X + 2$$

$$\therefore \quad y = -\frac{x}{3} + \frac{2}{3}$$

This is the same equation as ℓ_2 in Example 3. This shows that rotation of 90° about the origin is the same as the composite of reflection in line $y = x$ then reflection in the y-axis.

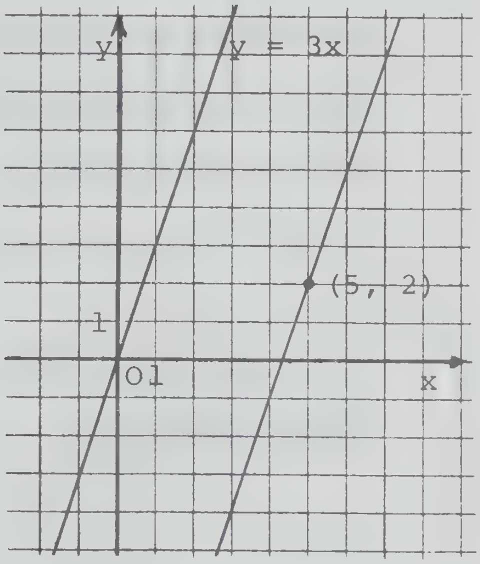
Composites of reflections are examined in more detail in 10Adv G 3b.

The method in Example 4 can be used to prove that two perpendicular lines have negative reciprocal slopes. Find the equation of the image of $y = mx + b$ under a rotation of 90° about the origin. The rule is $(x,y) \longrightarrow (-y,x)$. Thus $(X,Y) = (-y,x)$, or $x = Y$ and $y = -X$. The equation of the image is $-X = mY + b$;

$\therefore\ mY = -X + b;\ \therefore\ \ y = -\frac{1}{m}x + \frac{b}{m}$.

Example 5

Find the equation of the line through (5,2) parallel to line $y = 3x$.



The required line is the image of $y = 3x$ under $(x,y) \longrightarrow (x + 5,\ y + 2)$.

Let (X,Y) be any point in the image.

Then $x + 5 = X$ and $y + 2 = Y$

i.e. $\quad x = X - 5 \qquad y = Y - 2$

The equation of the required line is

$$Y - 2 = 3(X - 5)$$

$$\therefore \qquad y = 3x - 13$$

This topic can be extended by discussing one-way stretches of the form $(x,y) \longrightarrow (x,by)$ and $(x,y) \longrightarrow (ax,y)$, and two-way stretches of the form $(x,y) \longrightarrow (ax,by)$, and by examining images of the graphs of $y = x$, $y = x^2$, $x^2 + y^2 = 1$, and various rectilinear figures under i) stretches, ii) translations, iii) reflections, and iv) composites of i), ii) and iii) for numerical cases of the mappings.
See the notes for 9Adv G 2f.

For stretches, observe that slope changes, length of a segment changes, size of angles change, the shape of rectilinear figures change, etc.; but other properties such as collinearity, midpoint, and concurrency are invariant.

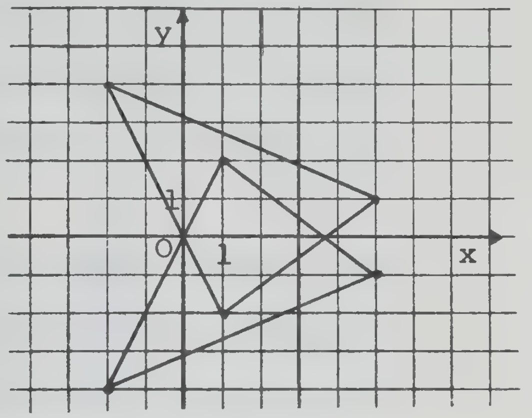
d) Using a 2 x 2 matrix to find images of points under certain mappings

This topic completely bridges the gap between geometry and numerical methods, as geometric figures may be transformed entirely by using numerical techniques with matrices.

If the student has been introduced to matrix multiplication in 10Adv N 4, then he could be introduced to this topic simply by multiplying a 2 x n matrix by a 2 x 2 matrix.
For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -2 \\ -1 & 2 & -4 \end{bmatrix}$$

Then the student could be told to plot the triangles with vertices (5,1), (1,-2), and (-2,4) and with vertices (5,-1), (1,2), and (-2,-4) to see the geometric significance.



The matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ can then be referred to as a matrix operator that acts on a matrix of points. It defines reflection in the x-axis.

On the other hand, if this is the student's first experience with matrix multiplication, it is probably better to first introduce the operator in relation to points in the diagram, one point at a time; for example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \begin{array}{l} \leftarrow\text{----} \text{row}_1 \times \text{column} \\ \leftarrow\text{----} \text{row}_2 \times \text{column} \end{array}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

This shows the student that each point is transformed onto its reflection image in the x-axis by matrix multiplication. Now draw the triangle with vertices the given three points, and the triangle with vertices the reflection images of these points.

Matrices can be written for the mapping rules of reflection, rotation, and dilatation studied in a). Matrices involve only numbers. The mapping rules of each transformation are easier to interpret and to recall. Matrices are easier to work with in more advanced problems, and can be entered in a computer.

The student can build the corresponding matrices for the mapping rules mentioned above. For example, reflection in $y = x$ is defined by $(x,y) \longrightarrow (y,x)$. Thus the 2 x 2 matrix must be such that

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

This is so if $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

is used.

A dilatation with centre the origin and scale factor 2 is given by $(x,y) \longrightarrow (2x,2y)$.

The matrix must be such that $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

This is so if $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

is used.

The student can be asked to draw image figures for various 2 x 2 matrices, to discover the nature of the matrix operator. Use the following matrices:

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix},$$

$$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}, \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix},$$

$\begin{bmatrix}-3 & 0\\0 & -3\end{bmatrix}$, and other dilatations, and optionally

$\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}, \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix},$ and other stretches.

The composites of two reflections could be investigated to show that it is the same as a rotation. For example:

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

In words, reflection in y = x followed by reflection in the y-axis is the same as rotation of 90° with centre the origin.

These can be investigated numerically and geometrically — they should produce a sense of 'awe' and give the student a taste of how the pieces of mathematics come together to produce a unified whole.

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GRADE 10 ADVANCED GEOMETRY

SECTION 3: ISOMETRIES

RELATED SECTIONS AND TOPICS

PAST	FY:	Pages 7, 12
	Ed PJ Div:	Pages 74-76
	Gr 7:	G 1; G 2; G 3; G 4
	Gr 8:	G 1; G 2; G 3ad; G 5df
	Gr 9 Gen:	G 1; G 2b; G 3
	Gr 9 Adv:	G 1; G 2be; G 3
PRESENT	Gr 10 Gen:	G 1; G 3
	Gr 10 Adv:	G 2; G 4; G 5

ISOMETRIES

Introduction

This section is intended to consolidate and extend the study of the congruence transformations (reflection, rotation, translation, and glide-reflection) that have been introduced in earlier courses. It serves as a basis for introducing transformation approaches to deductive proofs in 10Adv G 4. At this stage, the student should:

- be familiar with the fundamental properties of these transformations (see the notes for 7G 4b; 8G 2a; 9Adv G 3; 9Gen G 3);
- be able to construct the image of a figure under specific cases of each of these transformations (see the notes for 7G 4abc; 8G 2a; 9Gen G 2b; 9Gen G 3ab; 9Adv G 2b; 9Adv G 3);
- be aware of other properties of these transformations (see the notes for 8G 2a; 9Adv G 3acd);
- be able to identify corresponding parts of the object and image, and know that they are congruent (see the notes for 7G 4c; 9Adv G 3acde);
- be able to relate two congruent figures by one of these transformations and construct the defining property of the transformation (see the notes for 8G 2c; 9Adv G 3be);
- be familiar with real world situations which can be described by congruence transformations (see the notes for 8G 2bd);
- be familiar with line-, rotational-, and point-symmetry of plane figures, and recognize corresponding parts as being congruent (see the notes for 7G 1cde; 7G 4d; 7G 6e; 8G 1ac; 8G 2c; 8G 3ad; 9Gen G 1abc; 9Gen G 3a; 9Adv G 1; 9Adv G 3c).

a) <u>Glide reflection; properties</u>

See the notes for 9Adv G 3d.

b) <u>Isometries; invariant properties; direct and opposite congruence; determining the isometry for two congruent figures</u>

<u>Isometries</u>

The four congruence transformations are called isometries (from the Greek, 'isos' meaning <u>equal</u>, 'metron' meaning <u>measure</u>). They preserve the length of segments, and the size of angles. There are no other congruence transformations. This statement can be supported after the study of topics c) and d). Because of this, it is now possible to define the meaning of congruent plane figures in a way that applies to all figures, not just to special cases of triangles. "<u>Two plane figures are congruent, if and only if, they are related by an isometry</u>". Until this time, the student has tested figures for congruence by using tracing paper. If a tracing of one can be fitted on the other, then the figures are congruent. Now he/she has a mathematical definition that can be used in deductive proofs.

<u>Invariant Properties</u>

Under an isometry, many of the properties of the original figure are preserved in the image. These properties are called <u>invariant properties</u> of the particular isometry.

The student should investigate object-image relationships to compile a list of properties that are invariant for each isometry, and for isometries in general. A chart of the form below will help the student to summarize these properties.

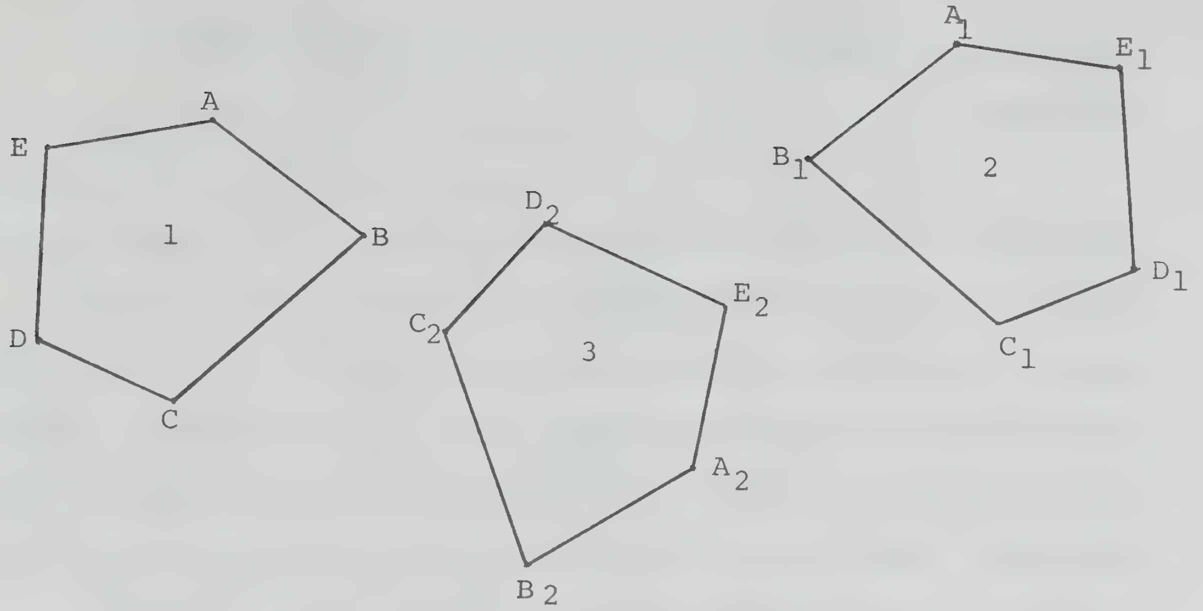
INVARIANT PROPERTIES

OF THE ISOMETRIES

	length	angle measure	congruence	slope	parallelism	perpendicularity	mid-point	point of division	sense (orientation)	collinearity	concurrency	tangency	circularity	area	shape (similarity)	size
reflection																
rotation																
translation																
glide-reflection																
isometry																
reflection in a point																

Direct and Opposite Congruence

If points in a closed figure (such as the vertices of a rectilinear figure) are assigned an order (in figure 1, from A to B to C etc. is clockwise), then the order of the image points A_1 to B_1 to C_1 etc. can be observed as the same (clockwise) or opposite (counter-clockwise). The order of the points is called the sense of the figure.



In the above figures, 1 and 2 have opposite sense; 1 and 3 have the same sense. The student might now be asked to compare the sense of 2 and 3 without looking at the figures themselves, and then to check this with the figures.

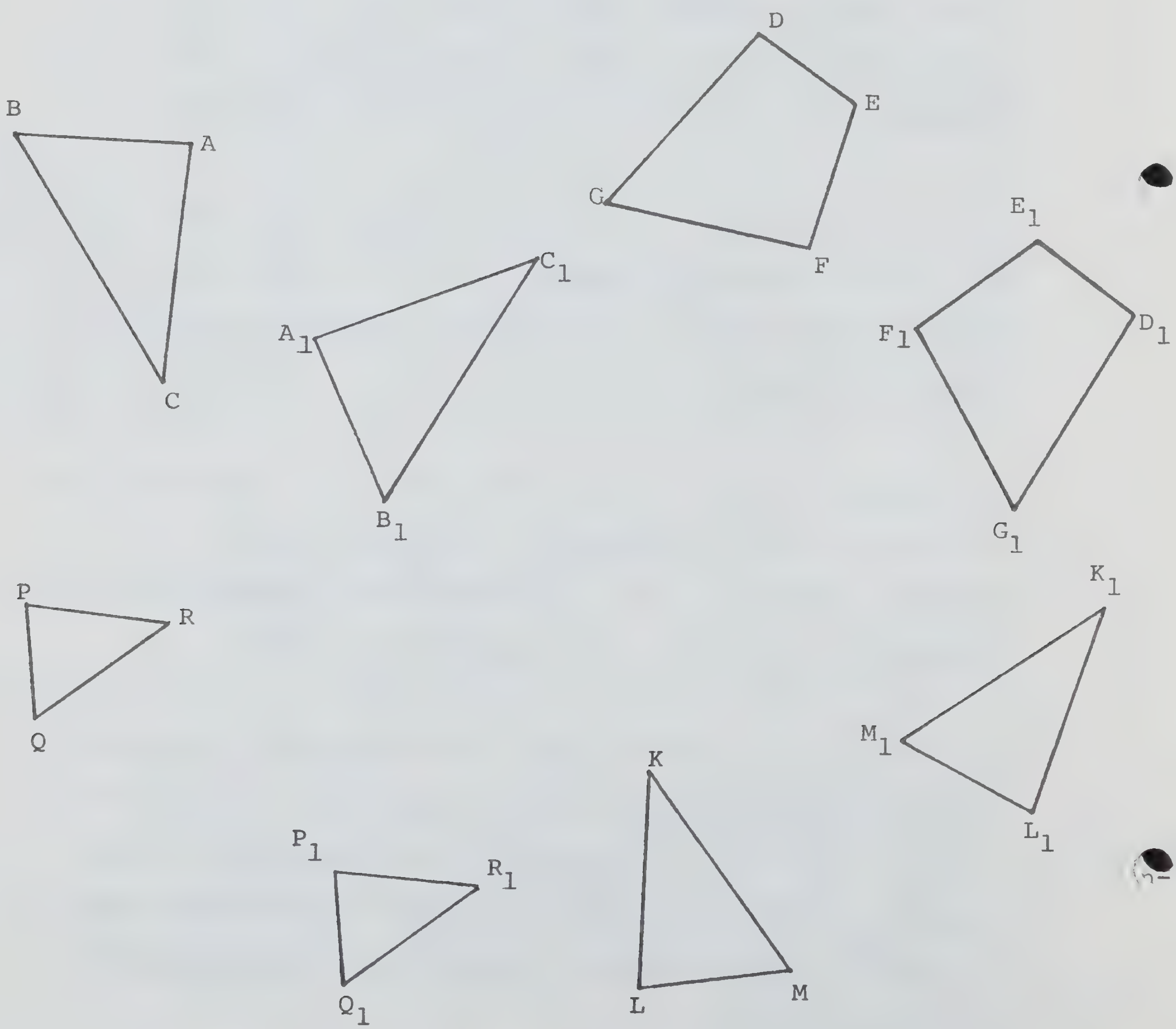
When two congruent closed figures have the same sense, we say they are <u>directly congruent</u> (1 and 3); when they have opposite sense, we say they are <u>oppositely congruent</u> (1 and 2, 2 and 3).

By observation and from the chart above, it is obvious that a figure and its <u>translation</u> or <u>rotation</u> image are <u>directly congruent</u>, and a figure and its <u>reflection</u> or <u>glide-reflection</u> image are <u>oppositely congruent</u>.

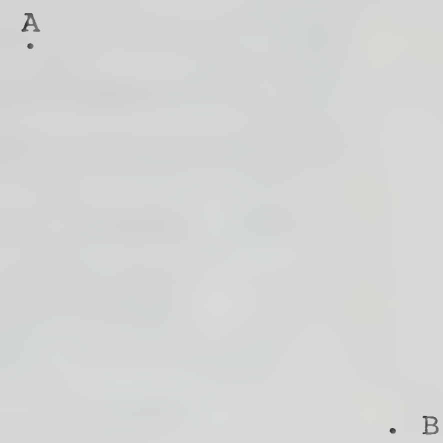
If a tracing of one figure will fit exactly on the other <u>without turning it over</u>, then the congruence is <u>direct</u> and the isometry is a <u>translation</u> or a <u>rotation</u>. If the tracing must be <u>turned over</u> to make a fit, then the congruence is <u>opposite</u> and the isometry is a <u>reflection</u> or a <u>glide-reflection</u>.

An activity similar to the one below could be used with students.

The pairs of figures below are congruent; check them for direct or opposite congruence. By observation of each pair, make a conjecture about the type of isometry. Now construct perpendicular bisectors of each pair of corresponding vertices, to determine the type of isometry and construct its defining property. Test the results with tracing paper or a transparent mirror (see the notes for 7 G 4b; 8 G 2ac; 9Adv G 3be).



It is interesting at this stage to investigate the minimum number of "object-image" pairs of points that are needed to define a given isometry. For example, if B is the image of A under a reflection then the reflection is uniquely determined; i.e. its reflection line can be drawn. This is not so if B is the image of A under a rotation. Explain.



How many object-image pairs of points are needed for a translation, a rotation, a glide-reflection? (Respectively, 1 pair, 3 pairs, 3 pairs.) For the latter two transformations, two pairs are often sufficient, but not in all cases. It is challenging to find the conditions under which the exceptions occur.

<u>Three Object-Image Points Determine the Type of Isometry</u>

Another interesting observation occurs when we know that we have an isometry, but we don't know which one it is. Only three pairs of object-image points are needed to determine the type of isometry. This is likely the reason why most authors of geometry books, when dealing with isometries, represent figures with triangles (any other polygon would add superfluous information).

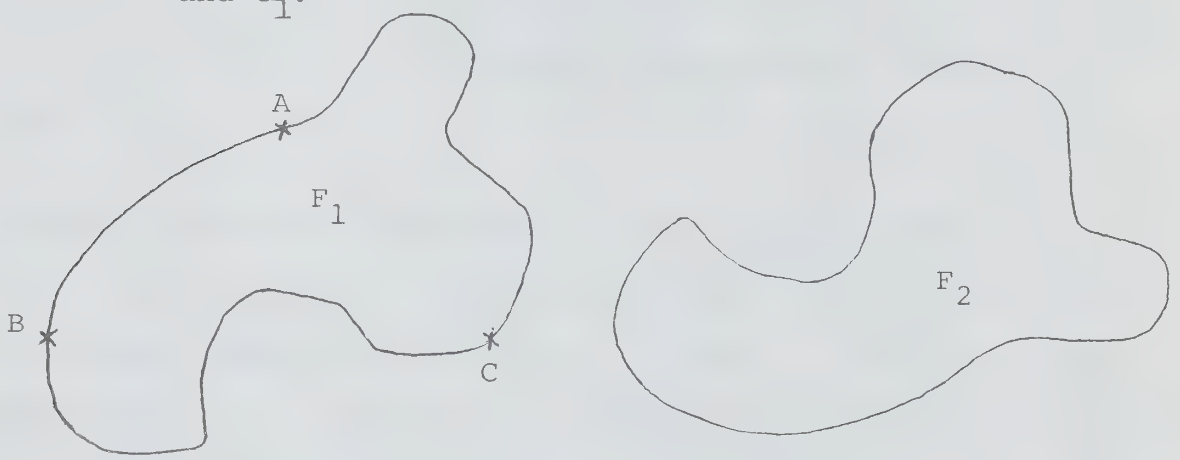
Given any two congruent figures, F_1 and F_2, the isometry by which they are related can be found by:

i) marking three non-collinear points in F_1: A, B, and C;

ii) making a tracing of F_1 and of A, B, and C;

iii) fitting the tracing exactly onto F_2 to locate the image points A_1, B_1, and C_1;

iv) finding the isometry that maps A, B, and C onto A_1, B_1, and C_1.

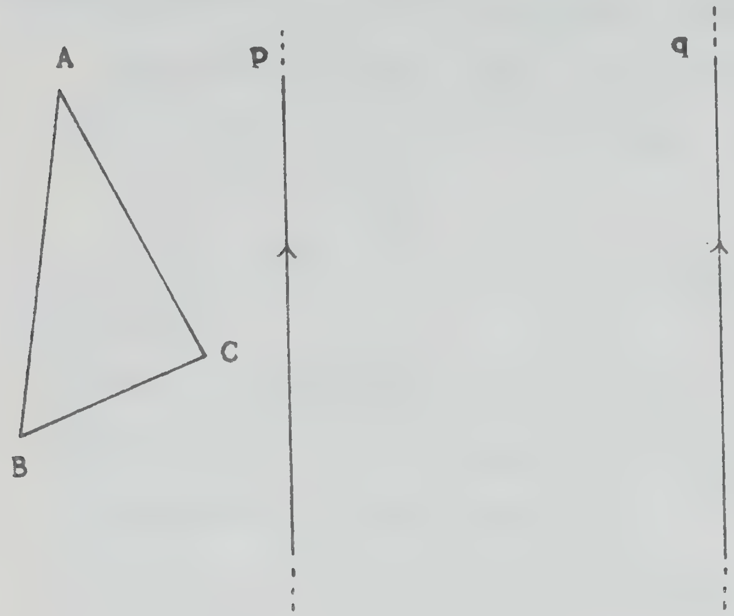


c) Successive reflections in two parallel lines as a translation, and in two intersecting lines as a rotation; fundamental properties of each

Two Parallel Reflection-Lines

Students should investigate the composition of reflections in two parallel lines. This means, for a diagram like the one below, that each student should reflect $\triangle$ ABC in line p,

then reflect the image in line q to locate $\triangle A_1B_1C_1$. Then the student should conjecture on how $\triangle ABC$ is related to $\triangle A_1B_1C_1$.



This investigation should be repeated a number of times, varying the distance between p and q, until the student conjectures that the composite of reflections in two parallel lines is a translation, with arrow twice the arrow <u>from</u> p <u>to</u> q. (The arrow from p to q must be perpendicular to p and q.)

The investigation can be extended by:

i) testing whether the order of the reflections in p and q affects the result (commutativity under composition?)

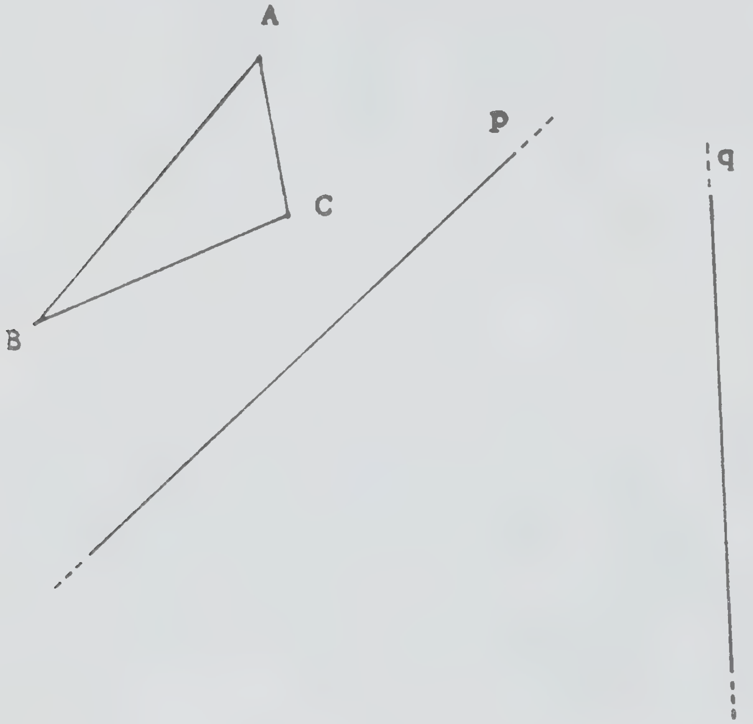
ii) testing how the location of p and q relative to $\triangle ABC$ affects the result (with p' and q' parallel to p and q and the distance from p' to q' equal to the distance from p to q.)

iii) testing the case for successive reflections in three parallel lines (consider associativity under composition)

iv) observing the direct and opposite congruences in the diagrams.

Save time by using grid paper to draw the parallel lines and images; some generality is lost, but this is not serious.

The students should also investigate the composition of reflections in two intersecting lines. Reflect in p, then reflect the image in q to locate $\triangle A_1B_1C_1$. Then the students should conjecture on how $\triangle ABC$ is related to $\triangle A_1B_1C_1$.



This investigation should be repeated a number of times, until the students conjecture that $\triangle ABC$ maps onto $\triangle A_1B_1C_1$ by a rotation, with centre the intersection of the reflection-lines, and measure twice the measure of the angle <u>from</u> p <u>to</u> q.

This investigation can be extended by

i) testing whether the order of reflections in p and q affects the result (commutativity under composition?);

ii) testing how the location of p and q relative to $\triangle ABC$ affects the result (with the angle formed by p and q constant, and the intersection of the lines fixed at the same point);

iii) testing the special case of composition of reflections in two perpendicular lines (commutativity under composition?);

iv) testing the case of successive reflections in 3 intersecting lines (consider associativity under composition);

v) observing the direct and opposite congruences.

It is interesting for the student to demonstrate how the figures are related by:

- making a tracing of Δ ABC, and
- pinning the tracing paper with a pencil point at the intersection of the lines, then
- turning the tracing to show that it will fit exactly onto $\Delta A_1B_1C_1$.

Thus we see that the composite of reflections in two parallel lines is the same as a translation with arrow twice the arrow from the first to the second line, and the composite of reflections in two intersecting lines is the same as a rotation with centre the intersection of the lines and measure twice the measure of the angle from the first to the second line.

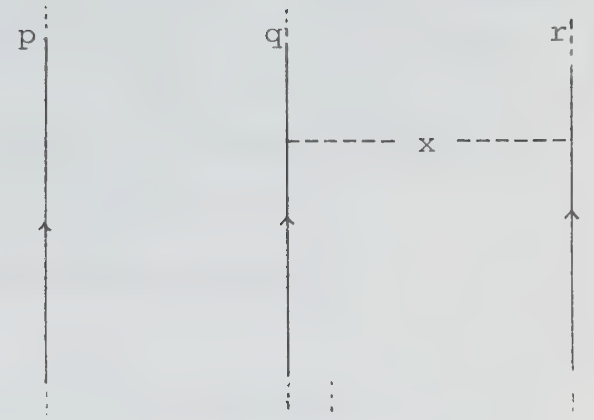
d) Composite of three reflections related to glide-reflection

i) Successive reflections in three parallel lines, three concurrent lines

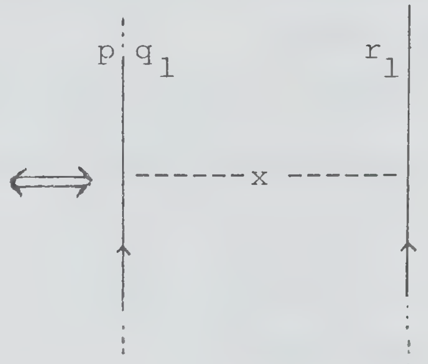
The composition of reflections in three parallel lines was suggested for investigation in c). The composite was discovered to be a reflection. It is interesting to explain why this is so. Similarly the composite of reflections in three concurrent lines is a reflection. Again the explanation is interesting.

For 3 parallel lines, consider the diagram on the right.

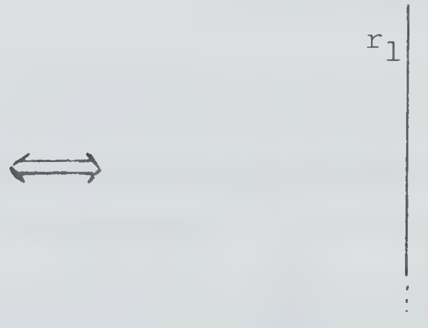
Reflection in p then q then r produces the same mapping as



reflection in p then q_1 (where $q_1 = p$) then r_1, as shown on the right. This is the same mapping as

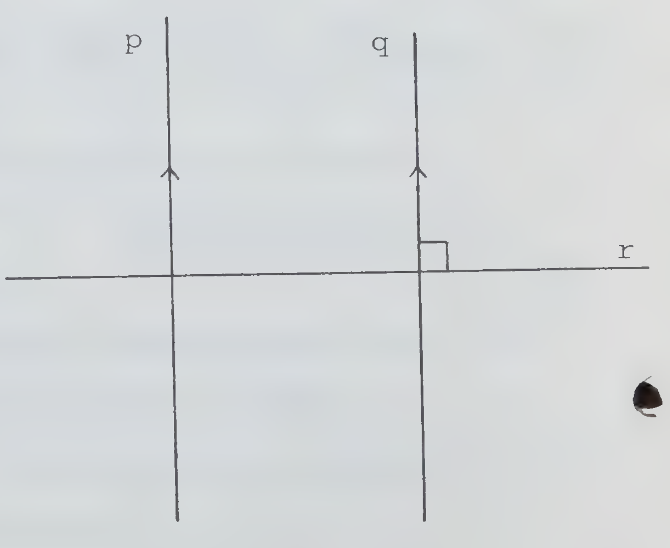


reflection in r_1.



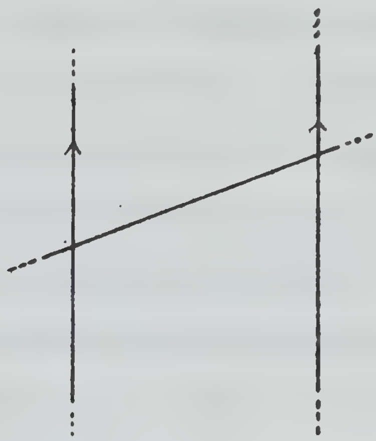
ii) <u>Two parallel lines, the third line is perpendicular to them</u>

The image can be drawn by the succession of reflections. Reflection successively in the two parallel lines is equivalent to a translation, this is followed by a reflection. This composite is a glide-reflection.



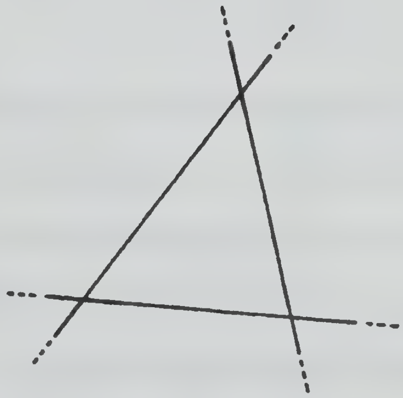
iii) <u>Two parallel lines, the third line is not perpendicular to them</u>

The image can be drawn by the succession of reflections. Then it can be shown to be a glide-reflection image by constructing the reflection-line of the glide-reflection.



iv) <u>The three lines are not parallel and not concurrent</u>

The image can be drawn by successive reflections; then the procedure of iii) can be followed.



We have dealt with all possible cases of the composition of reflections in 3 distinct lines, and have found the composite to be either a reflection or a glide-reflection. There are no other possibilities of reflections in 3 lines.

Composition and Composite

At this stage, a comment should be made to distinguish between the use of the words composition and composite.

Composition is used to describe a succession of mappings; for example, reflection in a line, then in a second line, and so on until the final image is determined. The single mapping that relates the object to the final image produced by the composition of mappings is called the composite of these mappings. There are infinitely many compositions that produce the same composite. For example, we have shown earlier that the composite of reflections in two parallel lines is a translation, and that there are infinitely many positions for the two parallel lines that produce the translation. See ii) on page 9. This is similar to saying there are infinitely many pairs of numbers that produce the same sum under addition.

A glide-reflection is defined by a composite of three reflections; that is, by a composite of a reflection and a translation parallel to the reflection-line (where the translation is the same as a composite of reflections in two lines perpendicular to the reflection-line). However, there are infinitely many compositions of three reflections that produce the same composite mapping -- any reflection then a particular rotation or translation.

Composition of two reflections is non-commutative. It would be meaningless to say that the composite of two reflections is non-commutative. Similarly, we say that addition of two numbers is commutative (3 + 5 = 5 + 3), we would never say that their sum is commutative (i.e., that 8 is commutative).

Only Four Isometries

Now we can show that there are only four isometries.

We have shown that directly congruent figures are either related by:

i) a translation (a composite of reflections in two parallel lines), or

ii) a rotation (a composite of reflections in two intersecting lines).

(Two distinct reflection-lines are either parallel or they intersect in one point; no other possibility exists.)

We have shown that oppositely congruent figures are either related by:

iii) a reflection (reflection, or a composite of reflections in 3 parallel lines or 3 concurrent lines), or

iv) a glide-reflection (a composite of reflections in 3 lines that are neither concurrent nor mutually parallel).

(No other possibility exists for 3 reflection-lines.)

By investigating the composition of reflections in 4 lines (or 6, or 8, or any even number), we see that the congruence of the object and the composite image must be direct and therefore must be a translation or a rotation. By investigating the composition of reflections in 5 lines (or 7, or 9, or any odd number), we see that the congruence of the object and the composite image must be opposite and therefore a reflection or a glide-reflection. There are no other possibilities.

We may re-define an isometry as follows.

An isometry is a reflection or a composite of not more than three reflections.

A composite of more than three reflections can always be reduced to a composite of three or fewer reflections.

e) Investigating other cases of the composition of two or more isometries

The student might investigate the composition of

i) two translations, to show that the composite is a translation;

ii) two rotations with the same centre, to show that the composite is a rotation with the same centre and measure equal to the algebraic sum of the other measures;

iii) a reflection and a rotation (or a translation) to show that the composite is a glide-reflection;

iv) two rotations with different centres, to show that the composite is a rotation or a translation (and investigate the special case when it is a translation);

v) a half turn about point A followed by a half turn about B, to show that the composite is always a translation defined by $2\overrightarrow{AB}$, regardless of the location of A and B;

vi) any two or more isometries, to show that the result is always an isometry.

Throughout these investigations, the student should make use of his/her knowledge of direct and opposite isometries to identify the composite as one of two possibilities. Further, the properties of closure, commutativity, identity, and inverse could be examined in each of these investigations. Associativity holds for the composition of any three isometries. If proof is to be considered for any of the conjectures made from specific cases, it is suggested that the translations, rotations, and glide-reflections be replaced by compositions of reflections, that associativity be used along with equivalent pairs of reflections to produce identity mappings ($r_\ell \circ r_\ell$ is an identity). This will permit the composition statement to be reduced to simpler form.

References

The following books contain interesting references related to isometries and composition of reflections.

- Geometry, An Investigative Approach; S.R. Clemens, P.G. O'Daffer; Addison Wesley (Canada) Ltd.; pages 156 - 218; teacher resource book

- Geometry, A Transformation Approach; A.F. Coxford, Z.P. Usiskin; Doubleday & Co. Inc.; pages 103 - 119; student and teacher resource book

- Geometry; H.R. Jacobs; W.H. Freeman and Co., San Francisco; pages 190 - 228; student and teacher resource book

Throughout these investigations, the student should make use of [illegible] of direct and opposite isometries to identify the [illegible]. Further, the properties of closure, commutativity, identity, and inverse could be [illegible] in each of these investigations. Associativity holds for the composition of any three isometries. If proof is to be [illegible] for any of the conjectures made [illegible], it is suggested that the translations, rotations, and reflections [illegible] be [illegible] to [illegible] the composition statement to be [illegible].

References

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Geometry, A Transformation Approach. Coxford, A.F., Usiskin, Z.P., Laidlaw Brothers [illegible], pages 103 - 113; student and teacher resource book.

Geometry [illegible] pages 138 - 150; student and teacher resource book.